Scattered Data Interpolation using Wavelet Trees

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Paper Review
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Interpolating wavelets

Building interpolation wavelets

- Classical interpolation tools can be fitted into a MRA framework.

Interpolating scaling function

φ is an interpolating scaling function $\Leftrightarrow \begin{cases} \varphi(0) &= 1 \\ \varphi(k) &= 0 \quad \forall\, k \in \mathbb{Z}^* \end{cases}$

- Built as infinite convolution of discrete filters $\widehat{\varphi}(\nu) = \prod_{k=1}^{+\infty} m_0(\frac{\nu}{2\pi})$
- Interpolating property can be expressed as $m_0(\nu) + m_0(\nu + \pi) = 1$
- φ has compact support $\Leftrightarrow h_n$ is finite $\Rightarrow$ Approximation order is finite
- **Families :** Splines functions, Deslauriers-Dubuc interpolating wavelets

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Lagrange interpolation :

- Can explicitly compute filter coefficients h_n with Lagrange polynomials of order p $\forall n \in [\![-2p+1, 2p-1]\!]$.

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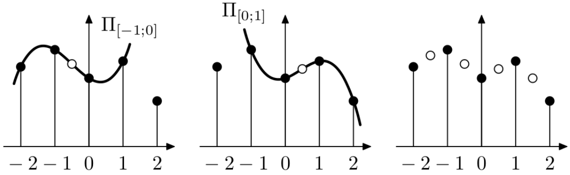
Dyadic refinement scheme :



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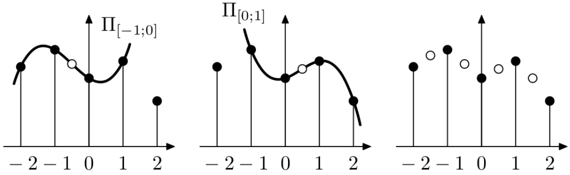


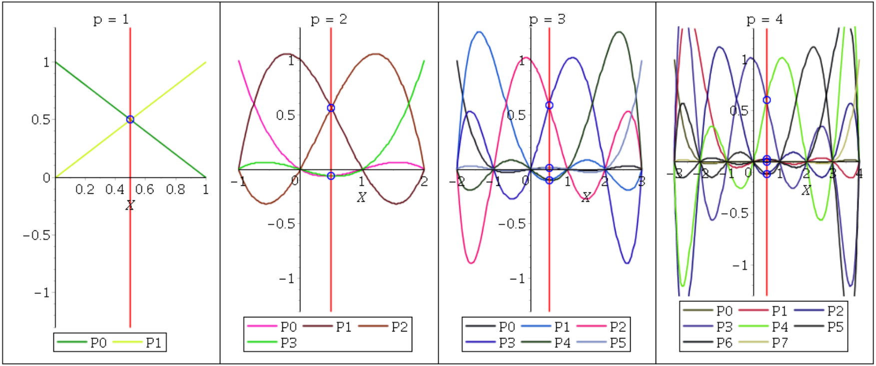
Low pass filter coefficients h_n

$$m_0[0] = 1 \qquad m_0[2k] = 0 \text{ if } k \in \mathbb{Z}^* \qquad m_0[k] = 0 \text{ if } |k| \geq 2p$$

$$m_0[\pm(2k-1)] = \frac{(-1)^{k+1}(2p)!^2}{2^{4p}p!^2(p-k)!(p+k-1)!(2k-1)} \quad \forall k \leq p$$

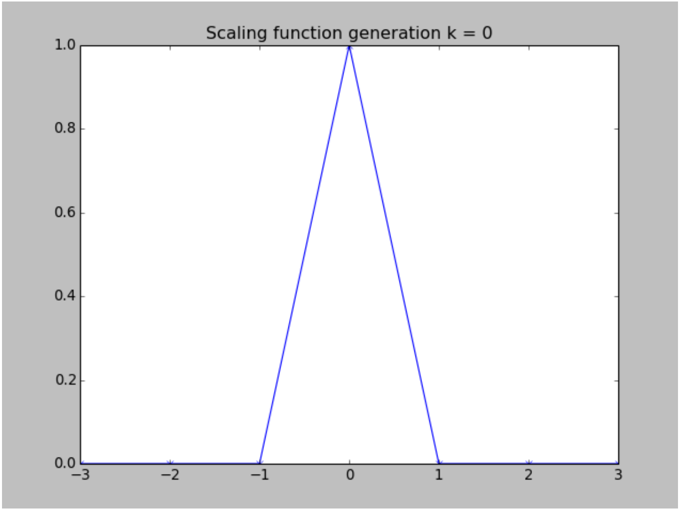
Generation of the coefficients

Just evaluate Lagrange polynomials to get the h_n :



Generation of the scaling function

Generate φ_p at resolution 2^{-j} with j convolutions $\delta_{x,0} * h_n^p * \cdots * h_n^p$:



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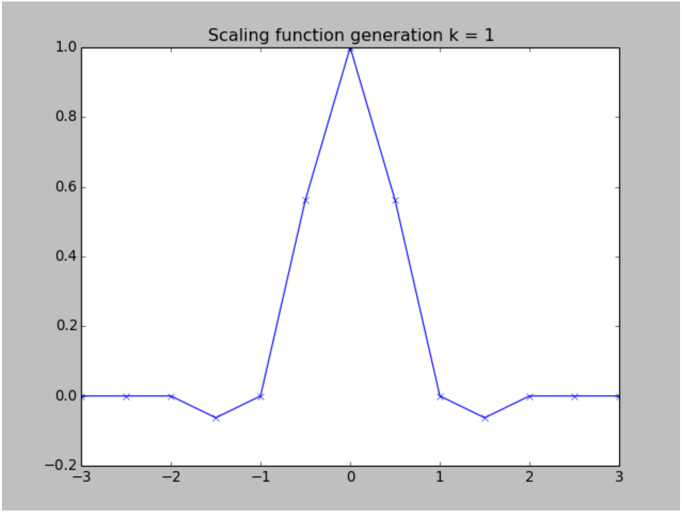

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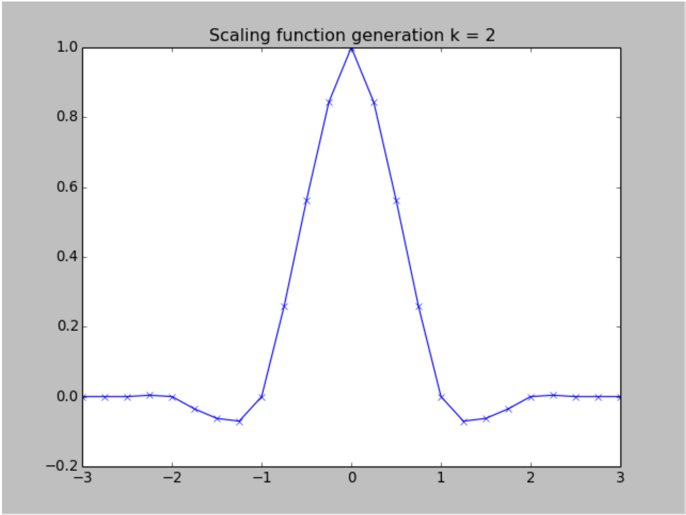

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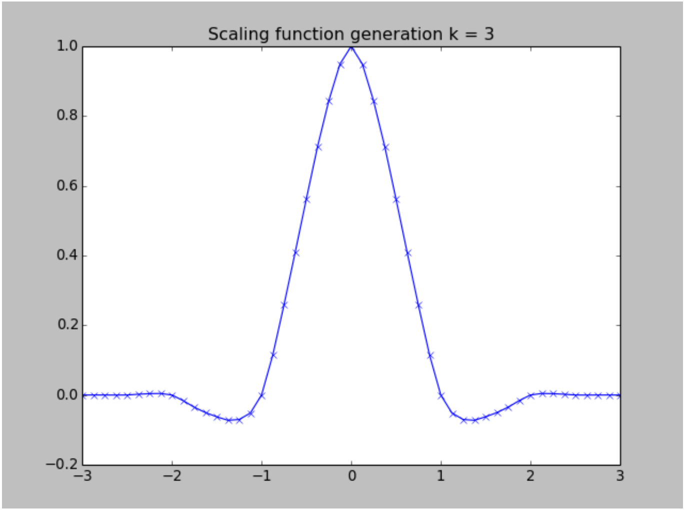

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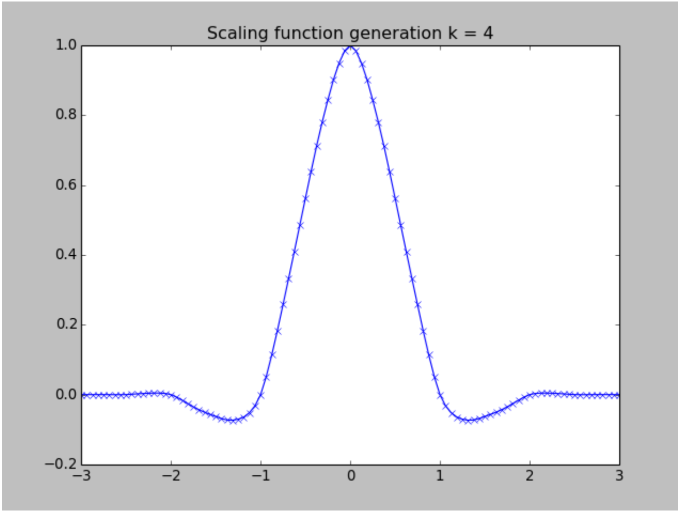

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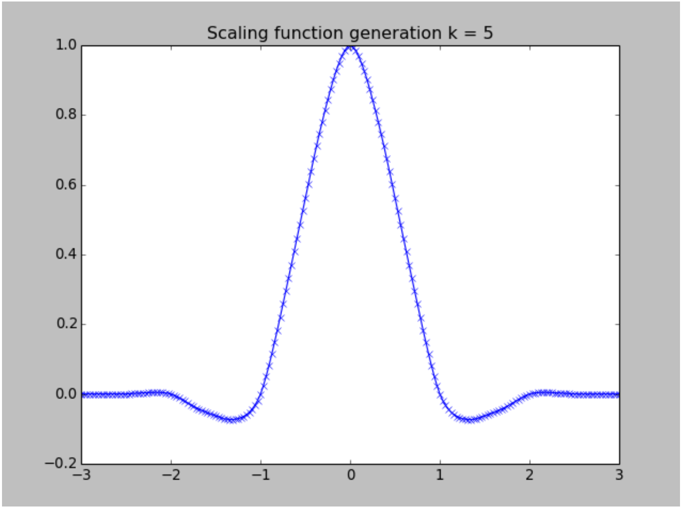

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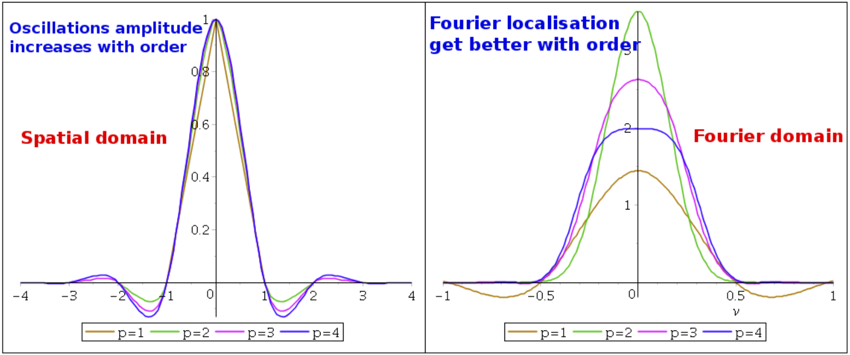

Properties

Basis definition in the dyadic case:

- Scaling function : φ(x) is of order p
- Mother wavelet : $\Psi(x) = \varphi(2x-1)$
- Wavelet family : $\Psi_{jk} = \Psi(2^j x - k)$
- Basis : $\mathcal{B}_0 = \{\Psi_{jk} \mid \underbrace{j=0}_{V_0} \text{ or } \underbrace{(j,k) \in \mathbb{N}^* \times (2\mathbb{Z}+1)}_{W_{j-1}}\}$

Properties :

- Symmetry
- Finite support $\subset [-2p+\frac{1}{2}, 2p-\frac{1}{2}]$
- No orthogonality $\Rightarrow$ need to solve a linear system
- $2p$ vanishing moments

Wavelets on the interval (boundary problems) :

- Take boundary filters of lower order

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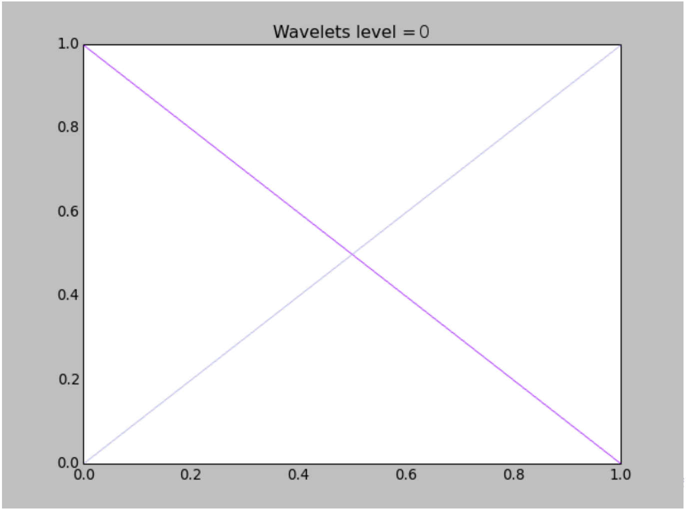
Wavelets on the interval : case $\Omega = [0, 1]$

Boundary filters : Lowest resolution => Highest resolution

V_0



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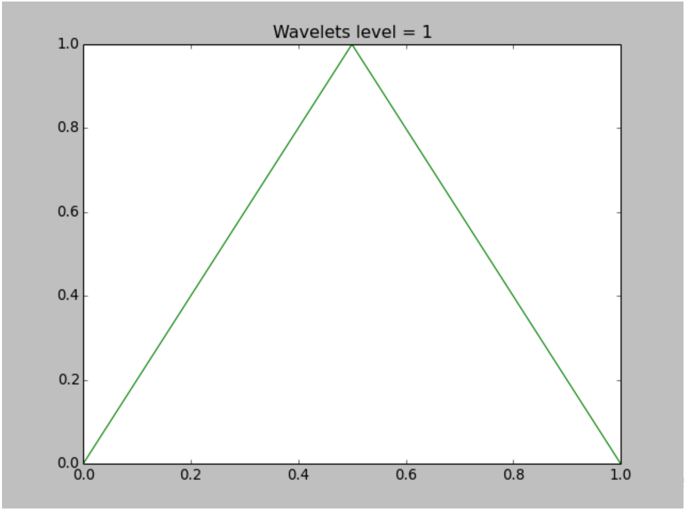
$$W_0$$



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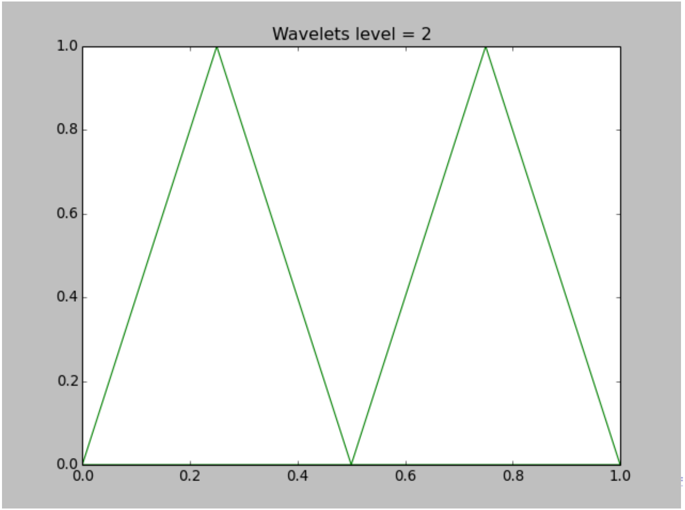

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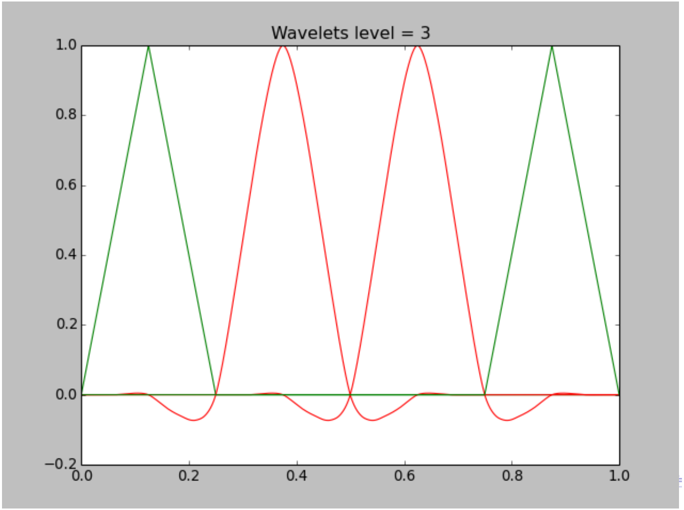

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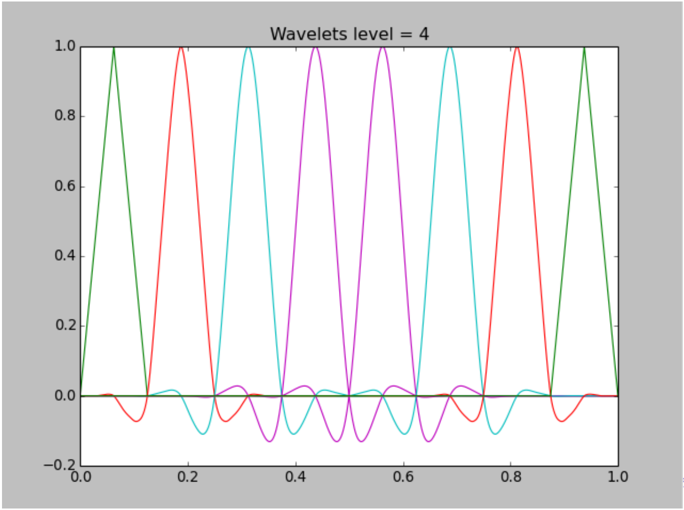
W_3



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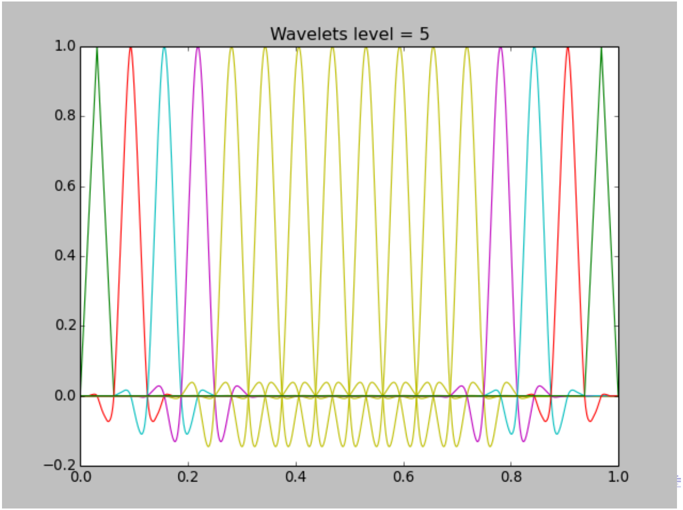
W_4



Wavelet tree approximation

Approximation in wavelet bases

Different approximations possible in wavelet bases :

- **Exact wavelet decomposition :** $f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle f \mid \Psi_{jk} \rangle \Psi_{jk} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} d_{jk} \Psi_{jk}$
- **Linear approximation** $\mathcal{A}^{\alpha}_{lin}$**:** $f \simeq \sum_{j=0}^{J} \sum_{k \in \mathbb{Z}} d_{jk} \Psi_{jk}$
- **Non-linear approximation** $\mathcal{A}^{\alpha}$ (best N-terms approximation) :

$$|d_{j_0,k_0}| \geq |d_{j_1,k_1}| \geq \cdots \geq |d_{j_{N-1},k_{N-1}}| \Rightarrow f \simeq \sum_{i=0}^{N} d_{j_i,k_i} \ \psi_{j_i,k_i}$$

- **Tree approximation** $\mathcal{T}^{\alpha}$: Build a wavelet tree with following relationship

$$(j,k) \ \mathcal{R} \ (j',k') \Leftrightarrow \begin{cases} j' & = & j+1 \\ k' & \in & \{2k-1, 2k+1\} \end{cases}$$

Tree approximation spaces are close to non-linear spaces (because singularities are located).

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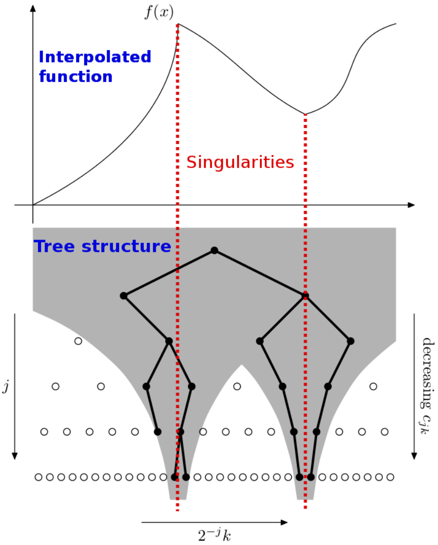
Example of wavelet tree structure



Scattered data interpolation using wavelet trees

Scattered data interpolation

Scattered data interpolation with wavelet trees:

- **Input:** Set of N samples $\mathcal{X} \subset \mathbb{R}$ with corresponding sample values $f_{\mathcal{X}} = \{f(x) \mid x \in \mathcal{X}\}$
- Samples are not aligned with wavelet centers $2^{-j}k$
- **Output:** $\mathcal{J}_S \subset (0, \mathbb{Z}) \cup \mathbb{N}^* \times (2\mathbb{Z}+1)$ and coefficients c_{jk} st.

$$\begin{cases} \forall\,(j',k') \in \mathbb{N}^* \times (2\mathbb{Z}+1) \quad \exists\,(j,k) \in \mathcal{J}_S \quad (j,k)\,\mathcal{R}\,(j',k') \\ f \simeq \sum_{(j,k)\in\mathcal{J}_S} c_{jk}\Psi_{jk} \end{cases}$$

Three step tree interpolation method:

- **Allocation :** Build one-one mapping from $\mathcal{X}$ to wavelet basis $\mathcal{B}_{\mathcal{J}}$
- **Subsystem selection :** Remove bad samples with a geometric criterion, new wavelet basis is $B_{\mathcal{J}_S} \subset \mathcal{B}_{\mathcal{J}}$
- **System solving:** Solve a linear system to find c_{ij} $\forall (i,j) \in \mathcal{J}_S$

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Three step tree interpolation method:

- **Allocation :** Build one-one mapping from $\mathcal{X}$ to wavelet basis $\mathcal{B}_{\mathcal{J}}$
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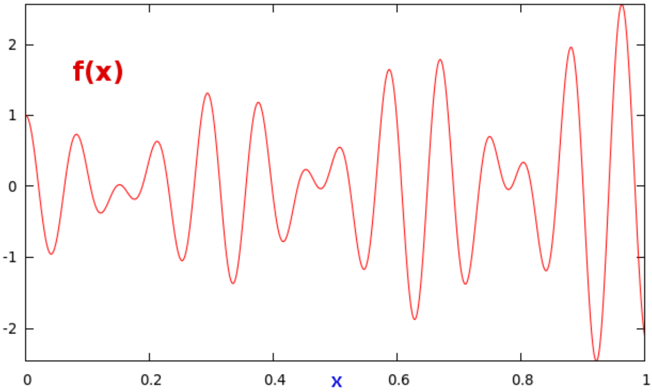
Example function

$\mathbf{f(x) = \cos(75x)\ e^x\ \cos(10x)}$ on $\Omega = [0, 1]$ approximated at order $p = 10$ with up to $N = 100$ uniformly distributed samples :



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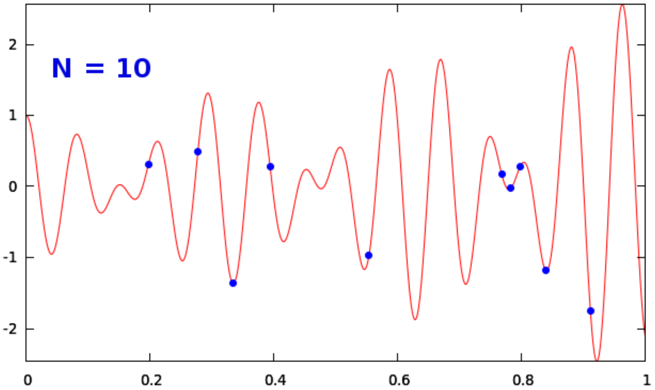

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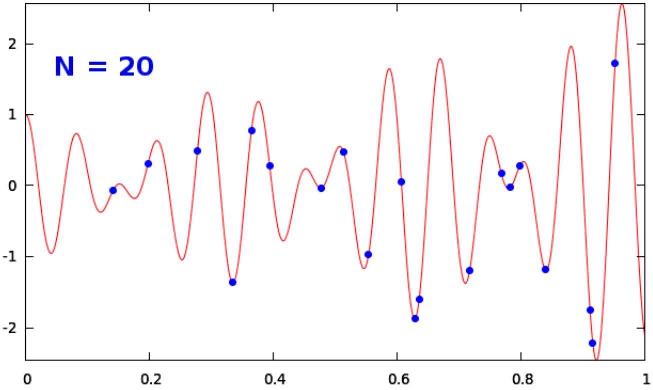

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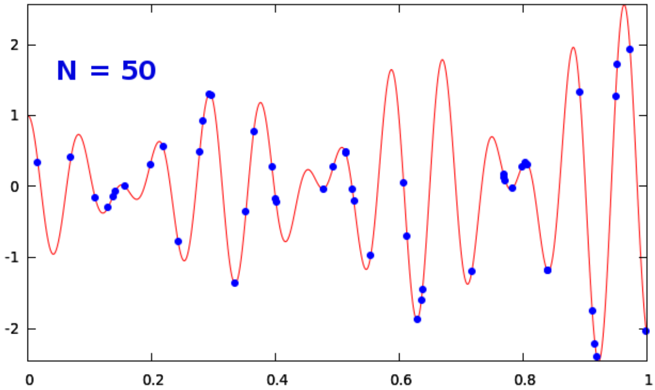

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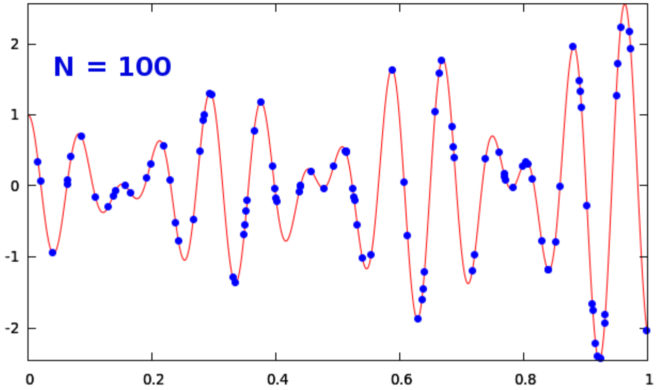

Allocation :

The goal is to select a wavelet subfamily to build an interpolating basis $\mathcal{B}_{\mathcal{J}} \subset \mathcal{B}_0$.

- Build a subfamily which provides a function that is a priori smooth $\Rightarrow$ Low resolution wavelets should be preferred.
- Select high resolution wavelets only where the density of measures is high.

Admissible allocation

$\mathcal{A} : \mathcal{X} \rightarrow \mathcal{J}_{\mathcal{A}} \subset (0, \mathbb{Z}) \cup \mathbb{N}^* \times (2\mathbb{Z}+1)$ is an admissible allocation $\Leftrightarrow$

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Order on the allocations

$\mathcal{A} \geq \mathcal{A}' \Leftrightarrow \exists j_0 \in \mathbb{N}$ such that

$$\begin{cases} \text{Allocations are the same up to level } j_0 \\ \forall k \in \mathbb{Z} \quad (j_0, k) \in \mathcal{J}_{\mathcal{A}} \Rightarrow |\mathcal{A}^{-1}(j_0, k) - \nu_{j_0,k}| \leq |\mathcal{A}'^{-1}(j_0, k) - \nu_{j_0,k}| \end{cases}$$

Theorem

Let $\mathcal{X}$ be a set of measure points. Then at least one of the following sentence is true:

- $\exists!$ optimal allocation $\mathcal{A}$
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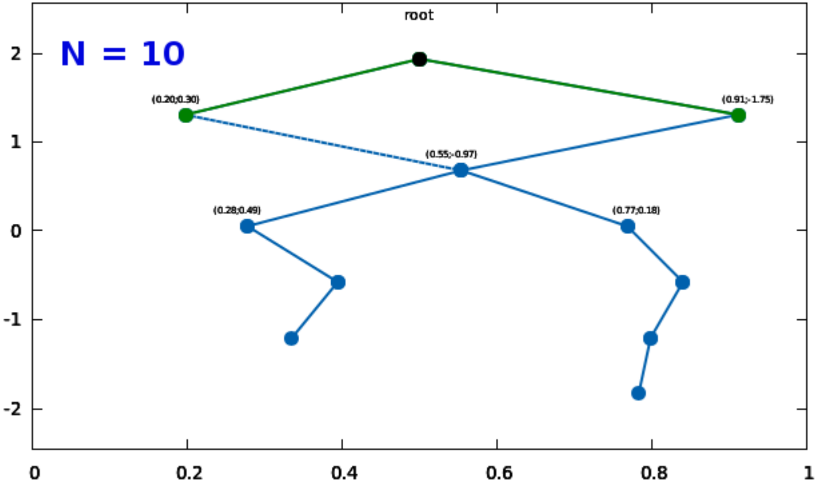

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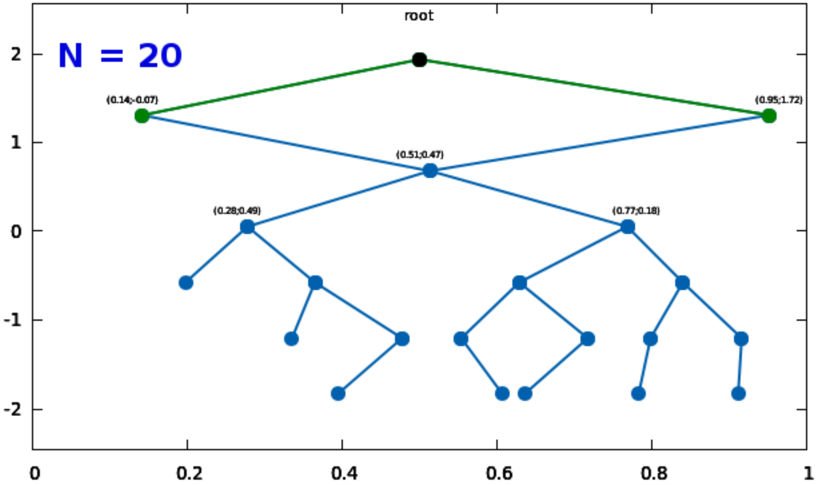

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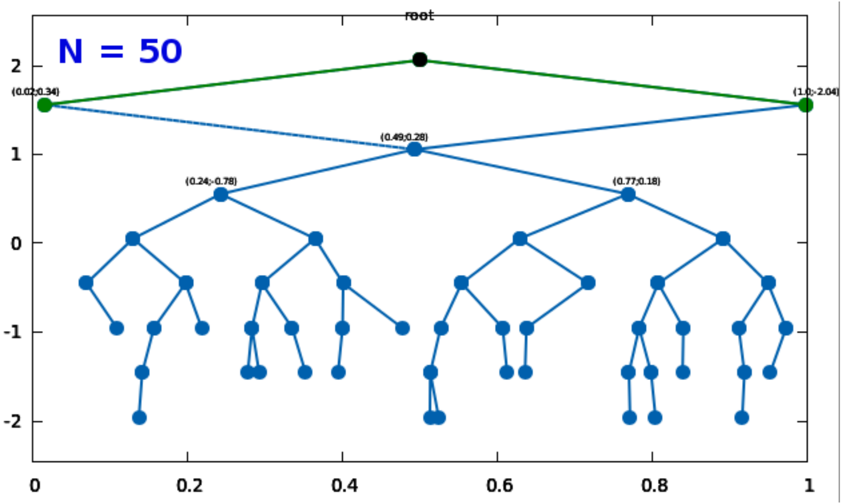

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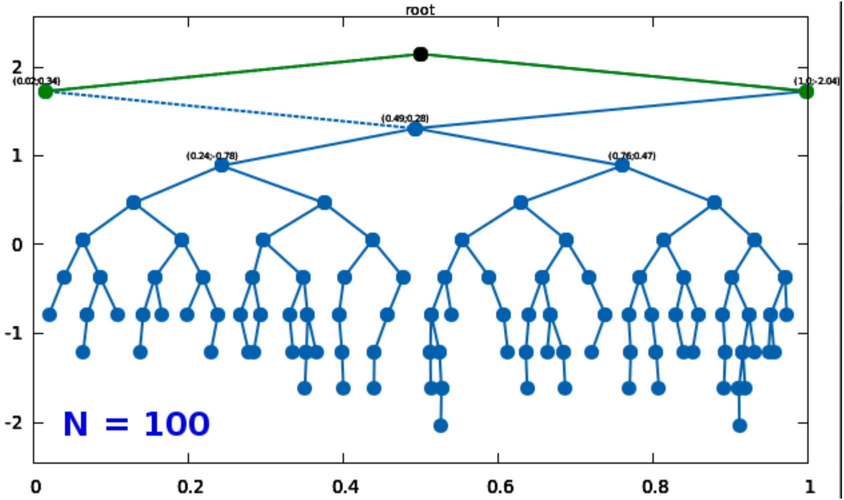

Second step : Subsystem selection

Problems entailed by allocation $\mathcal{A}$:

- Some points are badly located : too far from their wavelet centers.
- Linear system generated with previous allocation scheme may not be invertible.
- Need to extract the largest subtree $\mathcal{A}_S \subset \mathcal{A}$ such that we can compute the coefficients c_{ij} $\forall (i,j) \in \mathcal{J}_S$.

$\Rightarrow$ Just apply a geometric criterion to delete bad samples.

Exclusion criterion, placement condition

Let $(P,\rho) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\mathcal{A}_S$ fulfills the exclusion criterion of parameters (P,ρ)

$(j,k) = \mathcal{A}_S(x) \in \mathcal{J}_{\mathcal{A}_S} \Rightarrow$

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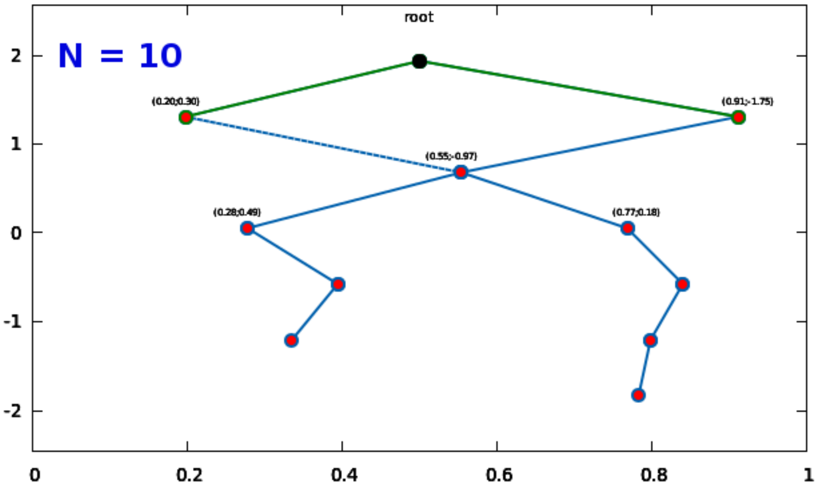

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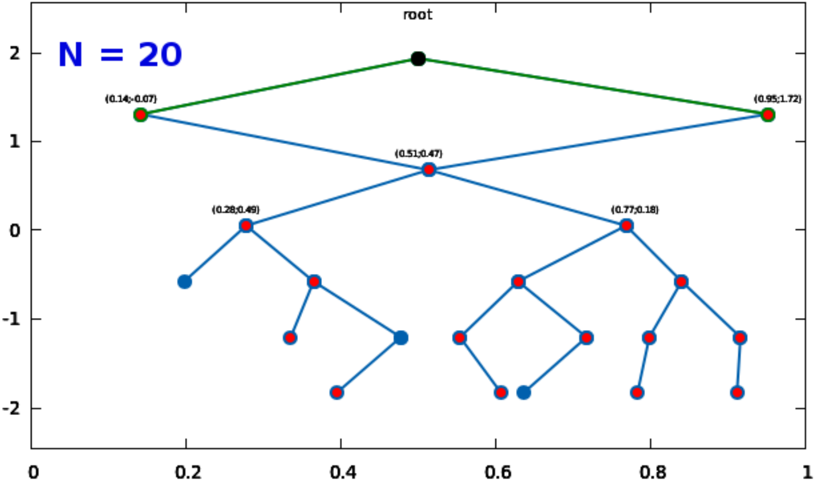

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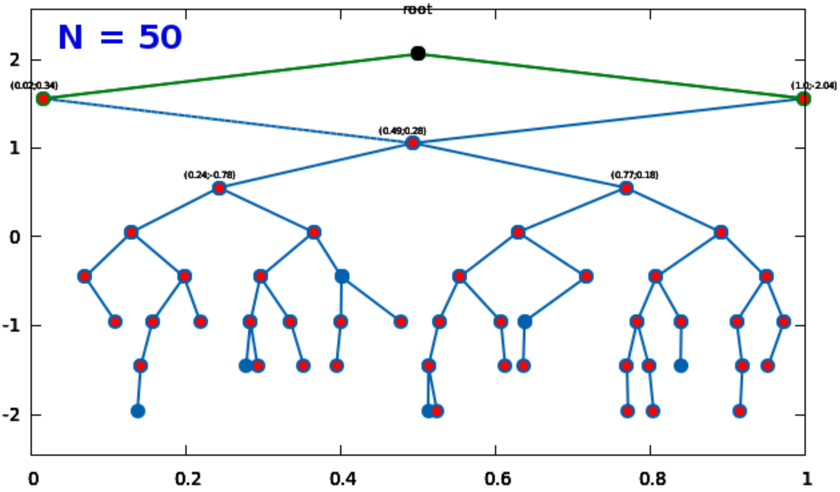

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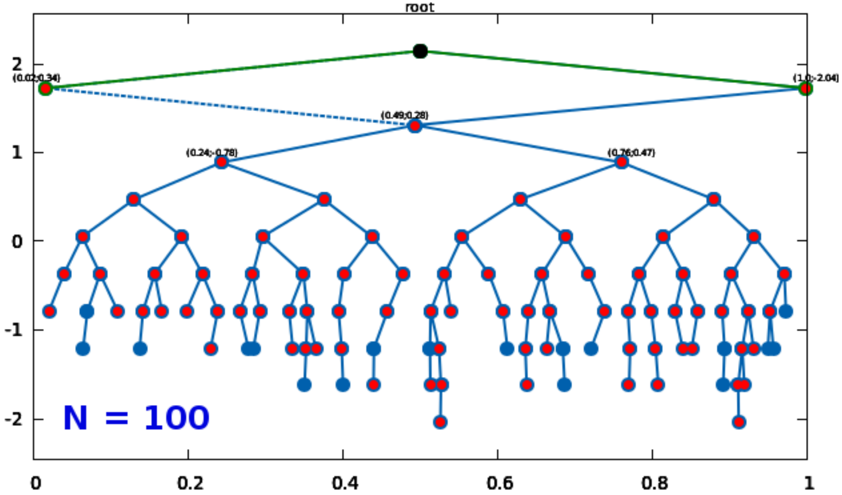

Last step : System solving

Finding the coefficients c_{ij} :

- Once bad points have been removed we obtain a smaller system.
- Solve square linear system of size N_S:

$$\sum_{(j,k)\in\mathcal{J}_S} c_{jk}\Psi_{jk}(x) = f(x)\ \forall x \in \mathcal{X}_S$$

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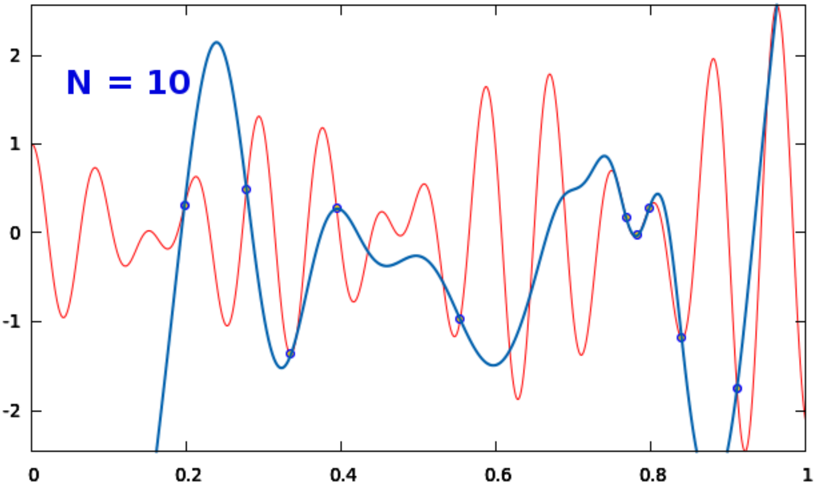
Results

Example of MRI at order $p = 10$



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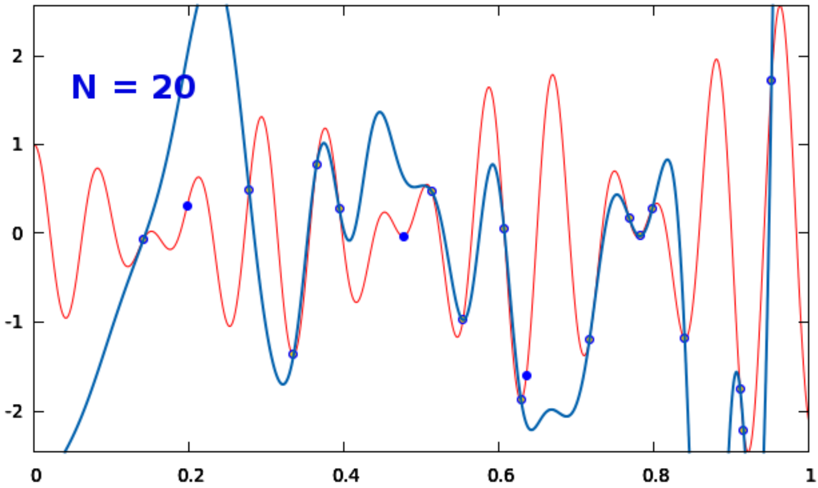

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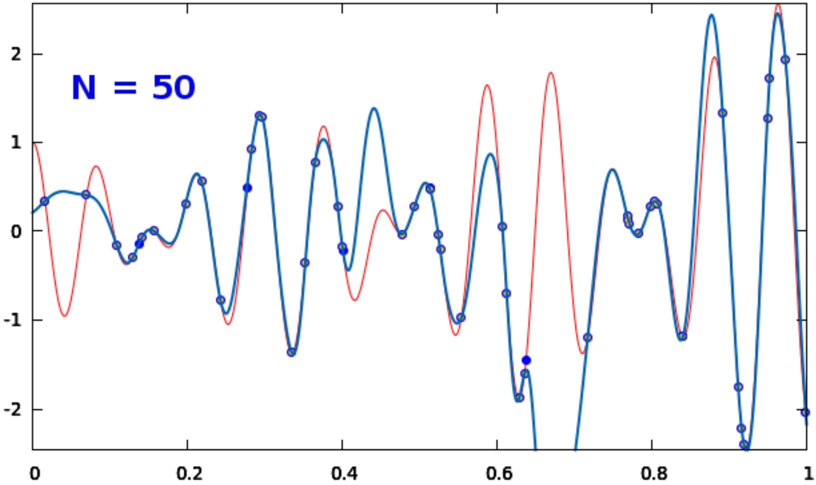

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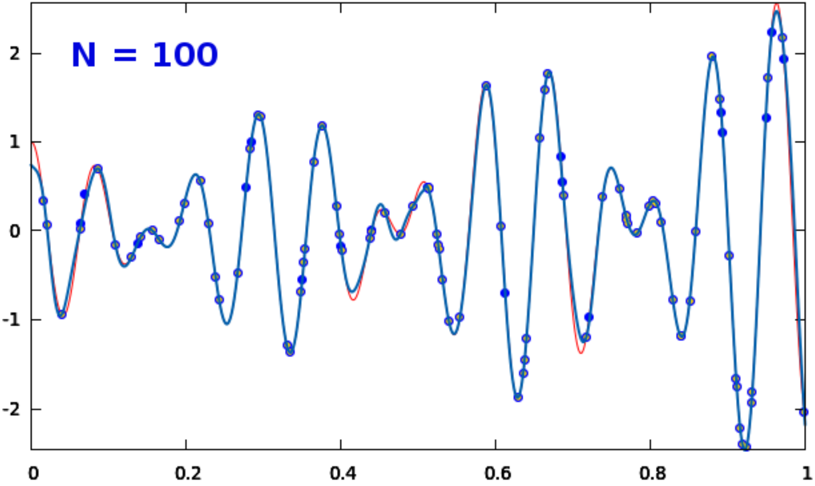
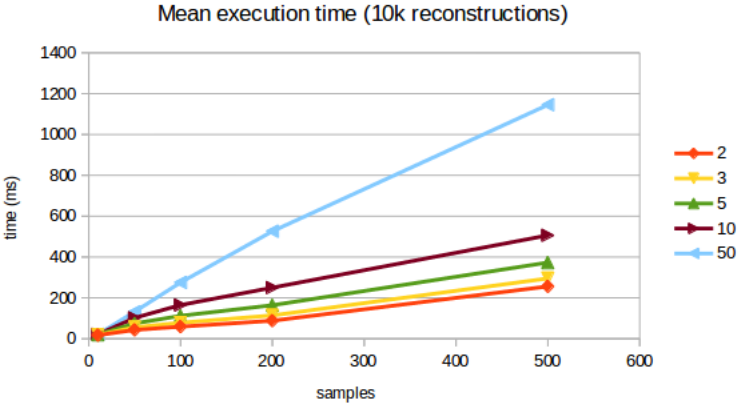


Error and performance versus order and number of samples



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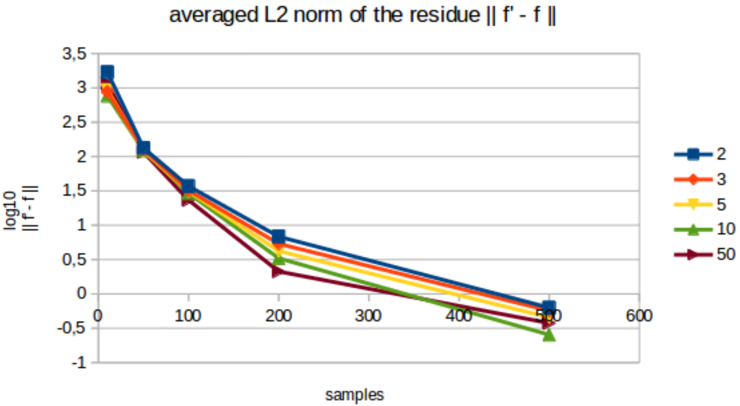

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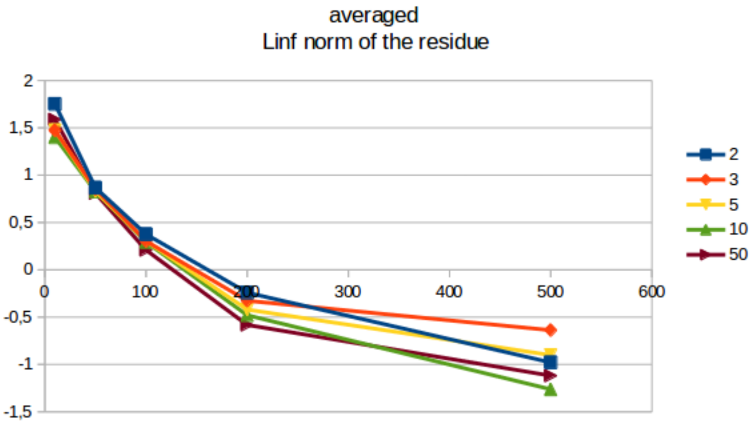

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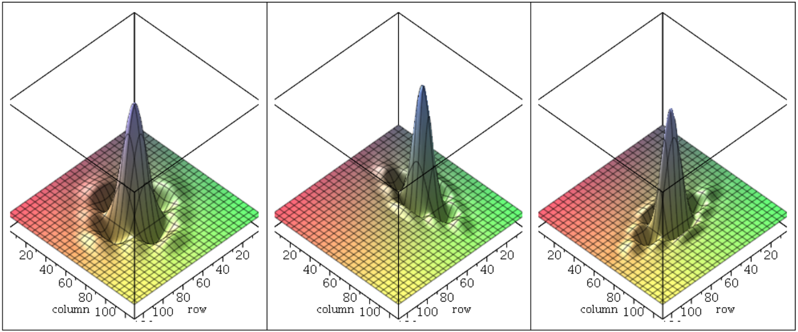

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- Incremental framework proposed, no need to solve the whole linear system at each sample added.

References

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Any questions ?